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Interactive Virtual Math: a tool to support self-construction graphs by dynamical relations

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An essential ability to understand mathematics and use it to solve problems is the ability to recognize, imagine and represent relations between quantities. In particular, covariation have been shown to be very challenging for students at all levels. The aim of the project Interactive Virtual Math (IVM) is to develop a visualization tool that supports students' learning of covariation graphs. In this paper we present the initial development of the tool and we discuss its main features based on the results of one preliminary study (before the development of the tool) and one exploratory study. The results suggest that the tool has potential to help students to focus on quantitative relations and relational reasoning. The results also point to the importance of developing tools that elicit and build upon students self-productions.

Keywords: visualization, Virtual Reality, interactive tool, secondary education

Introduction

The tool Interactive Virtual Math (IVM) can be found at https://virtualmath.hva.nl. The tool is designed to support 14-17 years old students at secondary school to understand the graphical representation of relations between variables in dynamic situations. IVM supports this process by addressing the visualization of these relationships. It is an interactive tool that allows students to draw, analyze and compare graphs for themselves and improve the graphs if they find it is needed. The version that we present at CERME 10 is a second prototype version of the tool in which we work with a single graphic situation. In later versions we expect it to be possible to use more contexts and varied assignments so that every student can practice at their own level. In Table 1 we present a short description of the main features of the tool.

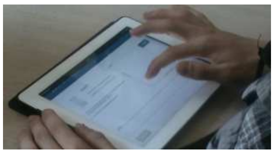
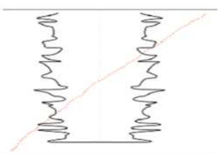
Theoretical framework

Students' difficulties with learning graphs are well documented in the literature. One type of difficulty is related with students' poor understanding of functions (Carlson, Oehrtman & Engelke, 2010). To model a dynamic situation (e.g. the speed variating with time or the height of water in a bottle variating with volume) into a graph, it has to be conceptualized as a covariation relation, that is a relationship between two variables that vary simultaneously (Thompson, 2011; Carlson et al., 2010). However, students have a tendency to view functions in terms of symbolic manipulations and procedures rather than as relationships of dependency between two variables. Students who don't conceptualize a function in this latter way will have difficulties imagining how the output values of a function are changing while imagining changes in functions' input values. Therefore they will lack the ability to construct a graph of a function modeling a dynamic situation. Another type of difficulty might be related to individual visualization abilities (Kozhevnikov, Kosslyn, & Shephard, 2005). Some students have the tendency to process visual information globally and this may hinder their ability to interpret and construct a graph by functional relations. According to

Hegarty and Kozhevnikov (1999), independent of a student's spatial ability, it might be possible for students to learn ways to develop such ability through instruction. Namely, instruction that encourages students to construct spatial representations of the relations between objects in a problem and discourage them from representing irrelevant pictorial details. The development of the IVM is inspired by these theories. This will be explained in the next section.

In Table 1 an example of the use of the tool and its main features are provided. The mathematical task used in this example is Task A from Figure 1, which concerns a dynamical situation involving the height of water in a bowl and the volume. To solve this task the students will need to consider how the dependent variable (height) changes while imagining changes in the independent variable (volume). The coordination of such changes requires the ability to represent and interpret relevant features in the shape of the graph (Carlson et al, 2010).

Table 1: main features of Interactive Virtual Math

Feature	Description
	***Self-construction*** The student is presented with two assignments. The first assignment is task A from Fig.1 and the second assignment is a variation of the same task with a cylinder instead of a bowl. In each assignment they are requested to draw a graph that describes the relationship between two variables in the corresponding dynamic situation. The student constructs the graph with a finger, a digital pen or mouse.
	***Contrast*** The student compares her or his own graph and explanation of two situations, referred to as *a* and *b*. The student can then submit the graphs or improve them.
	***Help 1*** The student visualizes the height of the water increasing in the bowl. He listens to the water he moves the platform with the ball and he can start and stop the water falling. Using a mobile device and a cardboard, Help 1 can be experienced as Virtual Reality
	***Help 2*** The student relates the graphical representation with the context representation. A Cartesian coordinate system in the plane and the bowl appear next to each other. The student must construct a dot graph that represents the height of the water in the Cartesian graph. He does this by dragging and dropping dots into the graph.
	***Reward*** The student gets the corresponding form of the bowl.

Many technological tools available for learning graphs involve visualizations but very few request students' own productions. They are often simulation-tools, which involve whole figures or part of figures that have to be moved, changed or dragged. When students are requested to construct a graph with these tools the construction means actually using representations that are given or are synthesized by putting parts together. In this case there is not a true visualization of students' concept image (Vinner, 1983) since part of the representation is already given. A distinguishing feature of the IVM is therefore that it builds solely on students' graphical productions. The graph must be drawn by the student themselves. The idea is to focus a student's attention on the relevant quantitative relationships, and engage them in the mental activity of visualizing these relationships without giving any graphical representation or part that has not been drawn by the student themselves.

Guiding principles and development of the tool

The development of the tool and the main features of the tool (*Self-construction, Contrast, Help 1 and Help 2, Reward* and *flow*) are based on general learning principles that include the importance of students' previous knowledge, interaction and feedback. We expect that the use of the tool elicit students to create their own graphs and explanations, to make assumptions, conjectures and to reflect upon these ones. The tool was also built according to topic specific learning principles. Thompson (2011) states that it is critical that students first engage in mental activity to visualize a situation and construct relevant quantitative relationships prior to determining formulas or graphs. Another guiding idea is to focus on visualizing the quantities. Results from Ellis (2007) indicate that instruction encouraging a focus on quantities supported generalizations about relationships, connections between situations, and dynamic phenomena. This idea is behind the features 'Help 1 'and 'Help 2' (Table 1). In Help 2 students must assume the height of the water in the bowl and represent it in the graph. Then they can drag the dot up to this height and observe the water coming into the bowl and compare the heights. We expect that the students, while guessing where to put the dot for the height, will notice that the difference between consecutives dots (values of the height) decreases in certain situations and increases in others. Another guiding principle was to provide constructive feedback to the students' final graph and to give them a way to evaluate their production. The student gets to see, after submitting his graph, the corresponding bowl-figure to the graph they draw. Finally, the tool also includes the use of Virtual Reality (VR), which is still limited to Help 1. Here the use of VR (sound, movement, interaction) is expected to improve the experience of the graphic situation.

To develop a tool we worked in a team composed by one researcher-math educator (first author), a high school teacher (second author) and ICT -developers. We focused on students aged 14-17 and we looked for an exemplary task with a graphic situation that was suitable to explore students' understanding of covariation and within a broad age group. We looked for a task in which the context is known of the students but the task self was not a routine task. The task should also reveal students' thinking and visualizations. We used task A (Fig.1), taken from Carlson et al (2010), who used it to diagnose students' understanding of graphs of dynamical situations.

Methodology

Preliminary study

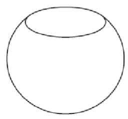
Previous to the development of the first version of the IVM tool we conducted a preliminary study to explore students' knowledge, skills and difficulties with making covariation graphs. In this first study (February-March 2016) that involved *N*=98 students from 4 classes age 15-17 years old, we used three versions of the same task with different questioning (Figure 1). The three questioning forms were: construct the graph given the figure of a bowl (task A); choose the correct graphic representation given the figure (task B); construct the bowl given the graph (task C). In the three situations the students were asked to explain their thoughts. The students in each of the four classes were divided into three groups and each group was presented with one of the three versions.

Figure 1: tasks used in preliminary study

Task A

Imagine this bowl filling with water.

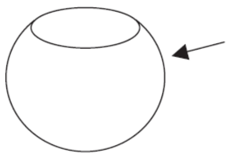
Sketch a graph of the water's height in the bowl as a function of the amount of water in the bowl.
Explain the thinking you used to construct your graph.



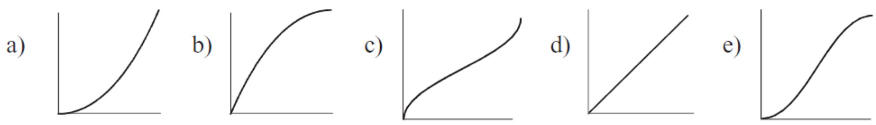
Task B

Assume that water is poured into a spherical bowl at a constant rate.

a) Which of the following graphs best represents the height of water in the bowl as a function of the amount of water in the bowl?
b) Explain the thinking you used to make your choice.

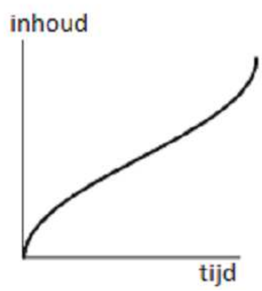


Spherical Bottle



Task C

Assume that water is poured into a bowl at a constant rate. The graph in the figure represents the height of water in the bowl as a function of the amount of water in the bowl. Describe the filling in of the bowl in words,



a) Explain the thinking you used to make the description.
b) Draw a possible bowl

Analyses of students' written answers showed that the majority of students that solved the self-construction tasks (tasks A and C) could not construct for themselves an acceptable representation (see Table 2).

Table 2: results of preliminary study

	Task A (self-construction graph)	Task B (multiple choice graph)	Task C (self-construction bowl)
Acceptable	12 (36%)	**25 (66%)**	3 (11%)
Incorrect	**19 (58%)**	11 (29%)	**22 (79%)**
No answer	2 (6%)	2 (5%)	1 (4%)

The majority of the students (64%) failed to successfully solve task A. Nineteen of them presented an increasing but incorrect graph, suggesting that they understand that the water increases or that the height increases with the amount of water but they don't have a consistent concept image of this process. Most of these students (13 out 19) produced one straight line (9 students) or a combination of two/three straight lines (4 students). It is interesting that 2 of the 12 students who have drawn an acceptable graph drew first a straight line and corrected it afterwards. These students seemed to have improved their solution either while providing an explanation or after drawing the graph. The results of this preliminary study were used to design the tool and to define its core features.

Exploratory study about the first version of the tool

The first version of the tool was developed in February –April 2016. We investigated the use of this first version in a follow-up study. We observed and interviewed four students with different school performance for mathematics age 14-15 years old (two boys and two girls) while working with the tool. Kevin[1] has high grades for mathematics, Lisa and Anton have average grades and Wilma has low grades. The aims of the exploratory study were:

- To understand how the students construct a graphical representation with the IVM.

- To identify features of the tool that support or constrain students' successful construction.

- To get a better understanding about how the guiding principles work and can be used to develop later versions of the tool.

The collected data consisted of video records and students' written work. The data was collected at two different moments in April 2016. In both situations the students were asked to go first through the whole application on their own. Lisa was the first student to be interviewed; she used the application on a computer. The other three students Kevin, Wilma and Anton were interviewed together at their school. Wilma and Anton use a tablet and Kevin a mobile device.

The data was first organized chronologically with relation to each students' attempt to construct the graph and use of the tool. Secondly, a global description of how each student attempted to construct

[1] The real names of the students were modified

and transform the graph was made and how they used the main features of the tool. A summary of the results are presented at Table 3. The results of these analyses and the data were shared and discussed with the ICT-team and used to evaluate the tool and to make decisions for the development of a later version. To get a better understanding about how the guiding principles work and can be used to develop later versions of the tool the researchers observed how students seemed to (or not) notice the relation between the variables; the relations between graphical and figural representations, and the reasons they gave when improving their constructions.

Results and Discussion

To understand how students construct a graphical representation with IVM and to identify the features of the tool that support or constrain students' successful construction we can take a closer look at Table 3. As we can see all four students improved their graphs on basis of different features. Each feature of the tool contributed to an improvement of the initial graph but, as Table 3 showed, different students used different features to improve their graph. This result suggests that students should be given the opportunity to choose whether they can view additional help or not and to be able to switch between the graphical situations. Furthermore, all students had difficulty with constructing a graph, even with the tool support. This result suggests that self-construction tasks are needed to reveal these difficulties, which can remain unnoticed when using simulation-tools or tools in which the representations are already given.

Table 3: students' use of the features of the tool during the exploratory study

Features	Kevin	Wilma	Anton	Lisa
Self construction (round bowl)	Acceptable final graph after two trials First trial produced incorrect graph with three straight lines and smooth corners Improved in second trial after reward	Acceptable final graph after two trials First trial produced two incorrect trials: a straight line followed by a parabola after seeing the second assignment	Incorrect final graph after two trials First trial produced several incorrect graphs (decreasing curves and switching between assignments one and two. Final graph is a curve raising slowly	Acceptable final graph after two trials
Self construction (cylinder bowl)	All students have produced an acceptable graph at first trial			
Contrast	First, all students draw a straight line at assignment one but improve their drawing after constructing the graph of assignment two.			
Help 1: Bowl is being filled up	Doesn't consult help 1 in first trial	Consults Help 1 and afterwards changes a straight line into a rising curve (still incorrect graph)	Consults Help 1 and afterwards changes the middle line of the graph, which is composed by three straight lines	Consults it but doesn't improve the graph
Help 2: Relation between figure	Doesn't consult Help 2 in first trial	Consults Help 2 and afterwards change a rising curve in an	Consults Help 2	Does not understand how it works

and graph		acceptable curve		
Reward	Uses it to improve his graph of assignment one: the straight line becomes a curve.	Not observed	Not observed	Does not understand the reward
Flow	Constructs both graphs without consulting Help 1 and 2.	Consults Help 1 and Help 2	Switching a lot between assignment one and two; and between Help 1 and assignment one	Consults Help 1 and Help 2
Virtual Reality (Help 1 with cardboard)	Not used	Not used	Not used	Mentions a more rich experience of the situation

In the case of one student (Anton) the improvement did not lead to a final acceptable solution. The student produced a graph with three straight lines and even after completing the two trials and listening to the explanation of the other students he remained in doubt whether the pieces of the graph should be curved or not. These results suggest that the tool has potential to support students to construct graphs by dynamical situations, but it is not on its own enough for some students. To understand the nature of the difficulties of Anton (and other students) we need to experiment more with the tool, also in combination with other tasks and forms of interaction (e.g., teacher feedback and working in peers).

We identified a number of aspects through which students could be brought to a relational understanding of graphical situations, while working with the tool. These aspects were observable and could be further explored in later investigations with the tool. These findings add understanding to why this tool can be of some value to mathematical education and, most of all, they can guide future iterations in the development of the tool. The actions that we observed were related to the focus on quantitative and relational reasoning.

Relational reasoning involves thinking about different representations, comparing them and relating them. We observed students' relational reasoning with the tool when:

- Students were comparing their own graph and bowl filling in with water (for instance Wilma when she used Help 1 or Anton switching from Help 1 to his own graph several times.)
- Students were evaluating the relation between the reward and initial graph (seeing the bowl of the reward set Kevin to think about the relation between the form of the bowl and the form of the graph. He used the reward to improve the smoothness of the graph curve.)
- Students contrast the relation between graphical situations of assignment one and two. For instance Anton switches between one and two; adaptation of graph one after seeing graph two.

Focus on the quantitative relation involves noticing that:

- When switching from assignment one to two they may notice that the relation between height and volume is different in the two assignments. That is for example the case of Wilma when she changes at assignment one a straight line into a rising curve.
- When observing Help 1 students may notice the relation between the increase of amount of water in the bowl and the different increase in height. For instance Wilma noticed by the filling of the

upper part of the bowl that when the same amount of water was poured in the increment in height differed.

- When using Help 2 students may notice the different rates in which the height increments in relation with the slices in the Bowl. So they can imagine the rate of growth as they have to decide the height of one point with relation to the previous one. For instance Anton explains that he has put the third point in the middle, almost the same increment in height as the previous point but a little bit less.

The students valued the opportunity of choice and the interactivity of Help 2 (one can drag and decide where to put the point). In the exploratory study only one student experimented with the VR aspects. Lisa valued the experience as a more enriching one. This is one of the aspects that we must investigate in future studies with the tool.

Concluding, this paper reports on the experiences of students learning graphical representations of covariation of related quantities with the aid of a new learning technology (IVM); a topic which many students struggle to understand. The small amount of students involved in the use of the IVM tool allowed for a fairly detailed study of their interaction with the tool. We have learned that the tool has potential to support students and we identified a number of aspects that could bring the students, while working with the tool, to a relational understanding of graphical situations. But, we also have learned that it is not on its own enough for some students. However, we should carefully interpret these findings since they regard only 4 students. We need to experiment more with the tool, also in combination with other tasks and forms of interaction to better understand its potential and in what extend these findings can be generalized.

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