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From speck to story: relating history of mathematics to the cognitive demand level of tasks

D. A. Agterberg¹ · R. J. Oostdam^{1,2} · F. J. J. M. Janssen³

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Abstract

It is a challenge for mathematics teachers to provide activities for their students at a high level of cognitive demand. In this article, we explore the possibilities that history of mathematics has to offer to meet this challenge. History of mathematics can be applied in mathematics education in different ways. We offer a framework for describing the appearances of history of mathematics in curriculum materials. This framework consists of four formats that are entitled speck, stamp, snippet, and story. Characteristic properties are named for each format, in terms of size, content, location, and function. The formats are related to four ascending levels of cognitive demand. We describe how these formats, together with design principles that are also derived from the history of mathematics, can be used to raise the cognitive level of existing tasks and design new tasks. The combination of formats, cognitive demand levels, and design principles is called the 4S-model. Finally, we advocate that this 4S-model can play a role in mathematics teacher training to enable prospective teachers to reach higher cognitive levels in their mathematics classrooms.

Keywords History of mathematics · Mathematical thinking · Cognitive demand · Teaching methodology · Teacher training

1 Introduction

Research shows that the majority of time in mathematics classes is spent on performing tasks (Roth & Givvin, 2008). A mathematical task is defined as a single complex problem or a set of problems focusing students' attention on a particular mathematical idea (Stein et al., 1996). The relationship between the nature of a task and the level of student thinking is a regularly recurring subject of research in mathematics education (Arbaugh & Brown, 2005).

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To describe the level of mathematical thinking of the tasks performed by students in mathematics class, Stein and Lane (1996) devised a four-level taxonomy. The first two levels are considered to be low in terms of cognitive demand, while the second two are defined as high cognitive demand activities. This taxonomy can be used to classify mathematical tasks (Stein & Smith, 1998) and has been used in practice and empirical research. Tasks can be coded for designed level of cognitive demand and compared with the enacted level of cognitive demand, like Kessler et al. (2015) have done.

According to Stein et al. (2007), curricula tend to rely on an accumulation of lower-level skills, before providing opportunities for higher-order thinking skills. Furthermore, the cognitive demand of a task can change throughout the implementation of the curriculum, from written to enacted. The cognitive level of high-level tasks is difficult to maintain (Henningsen & Stein, 1997). The level of cognitive demand of these tasks tends to decline during implementation by the mathematics teacher (Davis et al., 2016), while tasks that are set up at a low level are usually implemented as intended (Stein et al., 2007).

History of mathematics (HoM) can be considered as an effective tool to design lessons of a higher cognitive level. HoM enables the teacher to illustrate the development of underlying concepts that are related to routines and procedures. By presenting the historical context, the teacher creates the possibility to demonstrate mathematics in action instead of presenting frozen formulas. When a mathematical task is approached less as a reproductive activity, but more as a creative process and meaningful activity, the cognitive level of demand is higher. In that respect, HoM can be seen as a source of inspiration for mathematics teachers to make their lessons more meaningful and cognitive demanding for students.

Utilizing HoM in teaching can be justified not only by cognitive arguments, but by cultural and motivational arguments as well (Gulikers & Blom, 2001). HoM can help to develop a cultural approach. It may also contribute to mathematics not being a defined abstract subject for students, but a subject that has connections with many other subject areas and application contexts (Tzanakis et al., 2000). As such, the subject mathematics is viewed as dynamic and exploratory, rather than as a static structured system of facts, procedures, and concepts (Henningsen & Stein, 1997). HoM can help students to understand that mathematics is a human activity and that people have struggled with it. They become more aware that they are dealing with a subject that has been developed and applied by many over time to solve all kinds of issues.

Jankvist (2009) points to the importance of making a clear distinction between history as a *tool* and history as a *goal*. History can be used as a tool for assisting the learning of mathematics. This involves motivational and other more affective effects, but also the support of the actual learning of mathematics. HoM can for instance identify obstacles for learning mathematics and provide different viewpoints for teaching it. On the other hand, history can be a goal in itself. With this approach, HoM is “not a primary tool for learning mathematics better and more thoroughly....,” HoM “serves to illustrate other historical aspects of the discipline” (Jankvist, 2009, p. 239). As such HoM can be seen by teachers as a source of inspiration for making educational choices (Furinghetti, 2007). It is a powerful and inspiring tool for teachers to use historical examples to explain and illustrate how and why certain mathematical concepts and solutions have been developed.

Despite the potential of HoM, little research has been done into how it can be utilized to influence the mathematical level of thinking. The present article submits a model that combines aspects of existing frameworks for introducing HoM in the mathematics lessons classroom with the cognitive demand framework of Stein and Lane (1996). Within the so-called 4S-model, several formats of the presence of HoM in curriculum materials are described in relation to different levels of cognitive demand. Concrete examples of mathematical assignments, some of which come from Dutch mathematics textbooks for lower

secondary levels, are presented, to validate the adaptability of the model. These examples will also illustrate the different levels of cognitive demand. Finally, the possible application of the model in teacher training programs will be discussed, in order to arrive at the *design-oriented* approach we pursue for the educational practice of mathematics teachers.

2 Formats for presenting HoM

According to Tzanakis et al. (2000, p. 208), integrating HoM in the classroom can be accomplished in three different yet complementary ways. Complementary does not mean that these ways of integrating HoM are mutually exclusive and must not overlap. Rather, it means that they can supplement each other.

A first way is by presenting direct historical information (such as names and dates) in curriculum materials. A second way is learning mathematical topics by using a teaching methodology inspired by history. To devise such a methodology, crucial steps in the historical development of mathematical topics must be identified and reconstructed. For example, for teaching the abc-formula for solving quadratic equations, developments such as solving equations geometrically by ancient Babylonians and Persian scholars could serve as an inspiration, as well as the development of algebraic notation in Europe in the sixteenth and seventeenth centuries. A third way to integrate HoM in the classroom is to introduce students to a deeper awareness of mathematics. Activities that involve mathematical awareness should include aspects related to both the intrinsic and extrinsic nature of mathematical activities. Examples of intrinsic aspects are notation, terminology, the role of problems, and proofs. Examples of extrinsic aspects of mathematics are social and cultural aspects, such as the status of mathematics as a discipline during different periods of time, and relations to other disciplines such as sciences, arts, and humanities (Tzanakis et al., 2000).

Both Jankvist (2009, p. 247) and Tzanakis et al. distinguish between explicit, e.g., integrating direct historical information, and implicit use of HoM, namely historical approaches. With regard to format, their main focus is on the size of the packages, varying from snippets and illumination approaches to modules and history-based approaches. For our own categorization of ways that HoM can be integrated in curriculum materials or classroom activities, we designed four formats. The term *historical snippet* as introduced by Tzanakis et al. (2000, p. 214) was used as an anchor for designing the formats. As Tzanakis et al. state, historical snippets are pieces of historical information, incorporated in mathematical texts. The pieces can be categorized through characteristics such as size, content, location, and didactical function. We preserve the term *snippet* for the average size piece of historical information with substantial relation to the text in which it is incorporated. Next, we distinguished one format that is considerably more comprehensive than a snippet, and two formats that are more limited, regarding both size and content. Because the standard was set by the snippet, a word that starts with the letter S, we chose according terminology for each of the three new formats: *speck*, *stamp*, and *story*.

The term *speck* was chosen to indicate something very small. The notion *stamp* was inspired by Katz (1993) in which book the author uses many pictures of authentic stamps that feature personages and artifacts from the history of mathematics. It refers to the notion of a small, separate box of information, alongside the main text. The fourth format *story* relates to larger historical packages that transfers a story of mathematics.

Table 1 presents an overview of the four formats and lists corresponding properties of each.

A *speck* is a very small piece of text with historical information that can be presented in a rather arbitrary location in the curriculum material or a mathematics lesson. It contains

Table 1 Formats for presenting HoM broken down to size, content, location, and functions

Format	Speck	Stamp	Snippet	Story
Size	Very small	Small	Small to medium	A large package combining snippets, stamps, and/or specks as a coherent ensemble
Content	Text only Not or barely related with instructional text and/or mathematics assignment(s) in class	Text with or without a substantive related illustration(s) Not or barely related with instructional text and/or mathematics assignment(s) in class	Text with or without a substantive related illustration(s) Substantive relation with instructional text and/or mathematics assignment(s) in class	Text with or without substantive related illustrations Includes instructional text and/or mathematics assignment(s) in class Demarcated by specific subject matter component of mathematics curriculum
Location	Incorporated at an arbitrary location in curriculum material	Incorporated at an arbitrary location in curriculum material	Incorporated at a particular location in curriculum material	Incorporated at a particular location in curriculum material
Goal	Transfers historical aspects of the discipline	Transfers historical aspects of the discipline	Can transfer historical aspects of the discipline	Can transfer historical aspects of the discipline
Tool	Primarily affective, motivational	Primarily affective, motivational	Primarily support of learning of mathematics itself	Primarily support of learning of mathematics itself

for example information about the etymology of a mathematical word like parabola or a short reference to a historical fact such as the lifespan of Pythagoras. No processing assignment is given to apply the historical content. Its main purpose is to provide interesting information to students, simply nice facts to have knowledge of. This knowledge can contribute to motivation or other affective effects towards the subject of mathematics. A stamp is largely identical to the format of a speck but is of a larger size and can consist of text only or of a text in combination with a related illustration. For example, a stamp could contain a short biography of Descartes including a picture of him. This can be given alongside a text on coordinate geometry, but also alongside a text on algebraic notation. Similar to the speck, no processing assignment is given. We consider both speck and stamp as formats that transfer historical aspects of the discipline, but not as tools to assist the cognitive process of learning mathematics.

The snippet is different from both speck and stamp, because of its substantive and explicit relation with the instructional text or mathematics assignment. The location of the snippet is deliberately chosen because of its function within the learning process. When students learn to write equations with symbols such as x and $=$ and the notation for exponents, an annotated fragment of the work of Descartes can be incorporated to illustrate the pros and cons of certain rules of notation. HoM is primarily used as a tool. A snippet can be small, the size of a stamp, or somewhat larger, but it is restricted to a specific assignment or piece of instructional or theoretical information. When it comes to several assignments or entire topics, a collection of snippets (possibly combined with specks and stamps) becomes a story. A story is a coherent ensemble, demarcated by specific subject matter such as quadratic equations or platonic solids. In a story, HoM is primarily used as a tool, although the possibility of using HoM as a goal is clear. A story can cover a single class-room session up to an entire module of several lessons.

Before we can engage in a further description on the use and value of our formats, two remarks are in order. First, the approach described above is not limited solely to written curriculum materials. Oral instructions and exercises given by a teacher, as well as digital materials, can be applied to use HoM in the classroom. Second, it must be emphasized that the formal requirements that capture the four formats stated in Table 1 still allow for varying degrees of difficulty that can be presented within the different formats. In terms of cognitive processing, a stamp describing the biography of a mathematician can be simpler than a stamp explaining why some mathematics has been developed. Likewise a snippet and a story can vary in size as well as in level of difficulty, for instance due to the complexity of the language that is used. In the next sections, we will place our focus on the levels of mathematical cognitive demand that can be related to the formats, thus focusing on the perspective of history as a tool for learning mathematics.

3 HoM related to levels of mathematical cognitive demand

Stein and Lane (1996) introduced the cognitive demand framework to differentiate between mathematical tasks based on the level of mathematical thinking evoked by the task. Within the framework, four levels of cognitive demand are distinguished. At the lowest level, tasks are focused on reproduction without any relation to underlying concepts or meaning. A next level concerns algorithmic tasks in which students use known procedures. According

to the *Task Analysis Guide* by Stein et al. (2000), the use of a particular procedure is either “specifically called for or its use is evident based on prior instruction, experience or placement of the task” (Boston & Smith, 2009, p. 122). At a subsequent higher level, tasks appeal to insight into the underlying concepts and the use of multiple representations. The aim of these tasks is to develop meaning and deeper levels of understanding of mathematical concepts. Finally, at the highest level, tasks demand a non-algorithmic or complex and/or creative way of thinking. At this level, students are required “to explore and to understand the nature of mathematical concepts and relationships” (Boston & Smith, 2009, p. 122). Students’ self-monitoring and self-regulation for problem solving is a prerequisite at this level.

Table 2 lists all four levels of cognitive demands and gives a short definition of each level as well as an example for a classroom activity. The definitions in the middle column are derived from the Task Analysis Guide (Stein et al., 2000). All examples given in the third column are deliberately chosen from the same mathematical topic, to make clear the increase in mathematical thinking.

The examples given in Table 2 do not explicitly mention the use of HoM. But the activities that are listed for levels 2, 3, and 4 respectively can easily be related to HoM. A speck on the origin of the symbol π , or a stamp on the calculation of π , can be combined with an activity where students follow certain (historical) procedures. Snippets and stories can contain historical information on the search for the circumference and area of a circle, by the Egyptians, Babylonians, and Greeks, connecting different concepts and procedures, and adding to the understanding of mathematics as a useful tool for problem-solving.

Next, we combine the levels of cognitive demand with the four formats from the previous section. A speck and a stamp are related to the lowest cognitive level, since they contain direct historical information only and no processing assignment is given. The historical information can lead to more cognitive demand, but not as part of the process of learning mathematics itself. A snippet is integrated in a mathematical text or task at a particular location. It is related to a specific concept, an algorithm, procedure, or other form of mathematical activity. Therefore, the level of cognitive demand starts at level 2, but can possibly have level 3 or 4 as well. Since a story is defined as a coherent ensemble of snippets, stamps, and/or specks, its cognitive level starts at level 2. The level of cognitive demand can vary within the entire story, since the snippets it consists of can vary in level of mathematical thinking.

In addition to the formats, we can also combine two signature design-principles from HoM with the levels of cognitive demand. The four formats, four levels of cognitive demand, and two design-principles are put together in Table 3, which we will refer to as the 4S-model. According to Jahnke et al. (2000, p. 291): “the study of original sources is the most ambitious of way ..., but also one of the most rewarding....” Integrating original sources in the mathematics classroom requires special care from the teacher with regard to objectives and adequacy of the source. The authors mentioned list several didactical strategies that can be used when analyzing an original source, such as cognitive debates, translation, validation of reasoning, and comparison. Depending on the contents of a specific mathematical primary source, it can be used to obtain different levels of cognitive demand. A step-by-step description or procedure contained in a (translated) primary source can be considered as procedures without connections (level 2). Comparing a procedure that is written in a primary source to a formula or algorithm that

Table 2 Levels of cognitive demand, ordered from low (1) to high (4): definitions and examples

Level of demand	Definition	Example
1. Memorization	Students recall previously learned facts, rules, formulae, or definitions	Students recall the words diameter and circumference of a circle
2. Procedures without connections	Students follow previously demonstrated algorithms to solve problems without linking them to the underlying concepts, meaning, or understanding	Students calculate the circumference of a circle, using a formula that is handed to them by the teacher or textbook, which includes the unknown number pi
3. Procedures with connections	Students follow previously demonstrated algorithms to solve problems, but do so in a manner that maintains close connections to the underlying mathematical concepts. Procedures can be chosen, adjusted, and compared	Students are guided to start with a circle with a certain diameter (say 2 units). They calculate the circumference of both an inscribed regular hexagon and a circumscribed square (6 units and 8 units respectively). They conclude that the ratio of the circumference of a circle to its diameter is between 3 and 4. They can also be led to conclude it must be a constant ratio, and that constant can be approached even closer with measuring devices and/or polygons that approach the shape of the circle more closely
4. Doing mathematics	Students are required to solve problems that demand complex nonalgorithmic thinking for which they do not have a predetermined pathway	Students are asked to come up with an (approximate) formula for the circumference of a circle. They can do this by calculating by approximation the circumferences of multiple circles, by choosing and refining inscribed and circumscribed shapes that are known to them. They can also choose to measure circumferences with geometric software. They use their findings to come up with a relation between diameter (or radius) and circumference formula. They state this relation in a (syncopated) formula

students have learned from their textbook can be seen as procedures with connections (level 3). When students analyze a primary source, where some mathematics is stated without any justification or proof, they can be instructed to inquire if the mathematical statement is correct, if it is always correct, and how this can be proven. These activities are at the highest cognitive level of doing mathematics (level 4).

The use of original sources for designing classroom activities can be seen as a specific branch of a larger design-principle for mathematics education: the genetic principle. Tzanakis et al., (2000, pp. 208–209) phrase this as a “teaching approach inspired by history...,” where “...history may enter either explicitly or implicitly.” Jankvist (2009) annotates a “history-based approach” in which HoM serves as a source of inspiration for the design of teaching activities. The genetic principle was discussed in great depth by Schubring (1978). Schubring distinguishes between the historical genetic principle and the psychological genetic principle, closely following Toeplitz’ direct and indirect genetic principle (paper translated by Fried & Jahnke, 2015) and incorporating work from Freudenthal (1973). The historical genetic principle implies that the chronology and actual events in the history of mathematics are explicitly maintained and mentioned in the instruction and/or assignments. An example of this teaching strategy on the number π could be as follows. The teacher first deals with how the Egyptians and other ancient civilizations calculated the circumference and area of a circle. Then the instruction focuses on the ratio of the circumference of a circle to its diameter that was approximated by Archimedes. The final stage would be algebraic formulas, using the symbol π that was standardized by Euler. This entire sequence can be accomplished with or without the use of original source material.

The psychological genetic principle means that the historical development is used as the underlying basis for making didactical choices and does not have to be made explicit. For instance, the order in which the number concept is developed for students in textbooks can be chosen according to the historical development from natural numbers to fractions to the concept of zero and negative numbers, instead of using the logically ordered sets \mathbb{N} , \mathbb{Z} , \mathbb{Q} , and \mathbb{R} . Another example is the introduction of the concept of the square root of a number. The square root of two can be introduced with a field with given equal sides and the unknown length of the diagonal, without mentioning the Babylonians who did this over 3000 years ago or showing the actual clay tablet that contains this information. This concept can be demonstrated through an algorithm by the teacher, connecting procedures from algebra to geometry (level 3), or compared to other procedures in the textbook by the students (level 3) or be reconstructed by the students themselves (level 4). A teaching methodology inspired by history can be used on a larger scale as well, to construct entire series of instructions and assignments.

When using HoM to obtain a certain cognitive demand level of tasks, it should be borne in mind that historical content can be very difficult to understand, but not in terms of mathematical task processing. In the next sections, we will elaborate on raising the level of mathematical thinking.

Table 3 The 4S-model, relating formats of HoM to levels of cognitive demand and design principles (*H*, historical genetic principle; *P*, psychological genetic principle)

Format	Speck	Stamp	Snippet	Story
Memorization (1)	H	H		
Procedures without connections (2)			H	H
Procedures with connections (3)			H & P	H & P
Doing mathematics (4)			H & P	H & P

4 Improving cognitive demand by using HoM

Historical information presented in the format of a speck or stamp can be used as a starting point to design a snippet at a higher cognitive demand level. This actually implies an upgrade of the two formats at the lowest level by transforming them into a snippet. Such a transformation can be attuned to the specific mathematical topic at hand or aligned to given assignments in textbooks. At the same time, it is of course also possible to construct an entirely new snippet or even a story, based on information from resources, such as books on the history of mathematics. As defined in the previous section, a snippet with a piece of historical information is designed in combination with instruction and/or processing tasks. Depending on the subject content and the substantive relation with instructional text and/or mathematics assignment(s) for the students, the level of cognitive demand of a snippet can go from level 2 to either level 3 or 4. This section demonstrates, through concrete examples, three different ways that the four formats for HoM can be used to either adjust or design activities with a high(er) cognitive level. We will proceed with these examples in order of increasing level of cognitive demand, or otherwise formulated from left to right according to the columns in Table 3. Some examples are from Dutch textbooks for lower secondary education; others are from our own material collection. Some examples will include the use of primary sources and/or the genetic principle.

We start with upgrading existing specks and stamps in curriculum materials to a snippet. Figure 1 shows a stamp on the notation of plus, minus, and multiplication. This small piece of information can be found at a somewhat arbitrary location in a chapter on arithmetic or algebra. According to Table 1, this fragment, containing only factual historical information, is a stamp and therefore has cognitive level 1.

However, this piece of historical information presented in the format of a stamp has a lot of potential for interesting classroom conversation with students and/or mathematical activities for the students to engage in. There are plenty of options to transform this stamp into a snippet at several higher levels of cognitive demand. The teacher could raise the question of the benefits of a proper notation. They could demonstrate this by showing how notations have developed over time. As mathematics became more complex, the need for clear notation became apparent.

The teacher could use the original sources for an activity by giving a selection of historical notations to the students. The teacher can explain the old notation and ask the students to use this notation in an exercise. HoM is then used as a goal, and the cognitive demand level is 2. The teacher could also merely show the notation and ask the students to decipher it, comparing them to our current notation, maybe trying to put them in chronological order

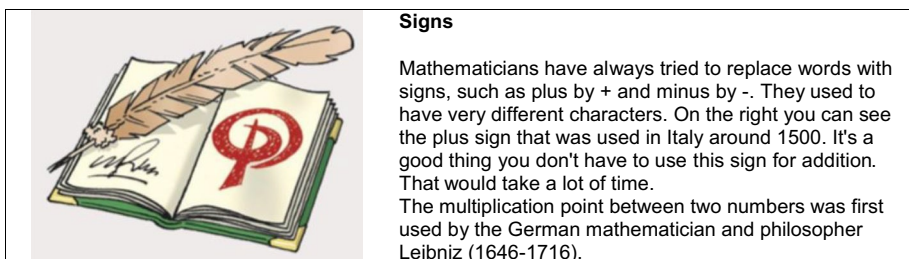


Fig. 1 An example of a stamp, taken and translated from the textbook Getal & Ruimte (2018, p. 58)

et cetera. Here we see HoM as a tool and the students are acting at cognitive level 3. To take this one step further, the students can be asked to come up with a new notation themselves, for instance before introducing them to exponents, have them consider options for writing $a \cdot a \cdot a \cdot a \cdot a$ in a short, unambiguous manner. Descartes, back in the seventeenth century, was the first to write this as a^5 . By having students create their own mathematical notation, and substantiating it, they are doing mathematics at level 4. As a final suggestion for an activity with this example of notation: why did this question of writing down long multiplications and the answer by Descartes come up this late in history, considering the age of a lot of other mathematical knowledge? This could be a very nice research project for students. HoM is used in this project both as a goal and a tool and the level of cognitive demand can vary from 2 to 4.

A second way to raise the level of mathematical thinking is by using the genetic principle for designing (or redesigning) snippets. We will give an example of both the historic genetic principle as well as the psychological genetic principle. The concept we use to demonstrate this is the process of solving quadratic equations.

Most textbooks approach quadratic equations in a strictly algebraic manner. After solving certain specific quadratic equations using Harriots principle (if $A \cdot B = 0$, therefore $A = 0$ and/or $B = 0$), the abc-formula is presented. The formula is followed by some worked examples and exercises that can all be handled with this same formula. In some higher-level textbooks, the technique of completing the square is also incorporated and possibly an algebraic proof of the abc-formula. When students learn that the abc-formula solves all quadratic equations (except when the quadratic polynomial's discriminant is less than zero), they will use it constantly, so they stay at cognitive level 2. When the teacher is able to have the students choose the most efficient procedure, the cognitive level is raised a bit, but it is still rather algorithmic.

When we take a historical approach with solving quadratic equations, it will immediately connect algebra and geometry. The term completing the square is actually originated from the geometrical surface of a square that through cut-and-past-geometry is constructed from another shape (a rectangle). This is the way the ancient Babylonians solved what we now call quadratic equations and the way the ninth-century Persian scholar Al-Khwarizmi solved these equations as well. His book *Al-kitāb al-mukhtasar fī hisāb al-ğabr wa'l-muqābala*, meaning *The Compendious Book on Calculation by Completion and Balancing*, has given us the word algebra we still use today: *al-ğabr*, meaning completing. Figure 2 shows a fragment from a textbook where students learn to solve an equation his way. Al-Khwarizmi's work is mentioned explicitly within a piece of subject matter, not in a separate frame alongside the text. In the exercises surrounding this fragment, the students are asked to use his method for solving quadratic equations.

Without going into detail about the full mathematical process, we limit ourselves to the claim that explaining how to solve quadratic equations through geometry before turning

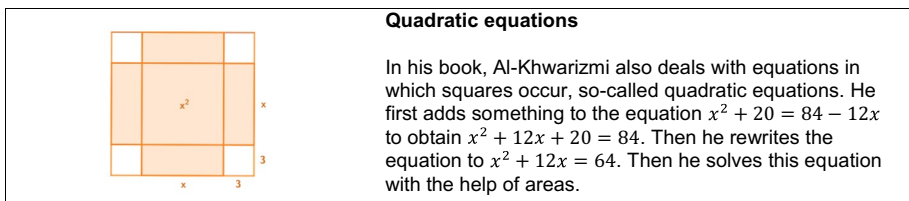


Fig. 2 An example of a part of a snippet, taken and translated from the textbook Kern (2019, p. 207)

to algebra, like it happened in history, will make this a level 3 or 4 activity, depending on the amount of discovery that is left to the students. If students learn to solve the equation according to Al-Khwarizmi (“*dixit Al-Khwarizmi,*” leading to the term algorithm), they will connect algebra and geometry, performing procedures with connections, at level 3. If the teacher (or the activity in the textbook) lets the students find out the geometrical solution themselves, they are doing mathematics at level 4.

The teacher can choose to use the historical genetic principle, by following the footsteps of the actual historical events and sharing names and dates while teaching how to solve quadratic equations. Another option for the teacher is to use the psychological genetic principle to design instructions and activities for students that use both the geometrical and algebraic insights from the past, in order to solve quadratic equations, without explicitly following or even mentioning the historical process.

Our third and final way to raise cognitive demand is by designing a collection of snippets, stamps, and specks on a coherent mathematical topic and structuring them into a story. In a way, our previous example on quadratic equations was starting to become a story itself. This depends on the size of the mathematical concept. Is it a single procedure or concept, then it can be presented as a single snippet. When we are dealing with a combination of concepts and procedures, a story makes more sense.

Stories can vary in size from a couple of pages to an entire chapter or book. In terms of lessons, a story could take up one session up to an entire module. We chose an example of a four-page story on platonic solids from a textbook. We show the last page in Fig. 3.


This story starts with the definition of platonic solids as regular shaped polyhedrons. Next it gives biographical information on Plato, who the solids are named after and who proved that there exist exactly five of these solids. The following exercises guide the students through this proof and lets them reconstruct a proof for themselves. Some more exercises are spent on discovering more properties of the solids, leading up to the rediscovering

10 a) Complete the table below

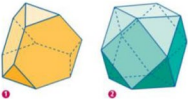
Solid	Shape of faces	Number of vertices	Number of faces	Number of edges
Tetrahedron	Triangle	4	4	4
Cube
Octahedron
Dodecahedron
Icosahedron

b) Check the following rule for the tetrahedron:
 “The number of vertices plus the number of faces minus the number of edges equals two”

c) Check this rule for the other solids in the table.
 The rule: “The number of vertices plus the number of faces minus the number of edges equals two” is known as Euler’s formula. This formula is shown on the stamp on the right. The German word for vertex is Ecke, the word for edge is Kante and the word for face is Fläche. Leonard Euler lived in the eighteenth century as one of the greatest mathematicians of all time.



11 Check that the formula $e - k + f = 2$ on the stamp corresponds to the rule of exercise 10b.



12 These solids are called half-regular.
a) Why do you think they are called half-regular?
b) Does Euler’s formula apply on these solids?

Fig. 3 An example of a part of a story, taken and translated from the textbook *Moderne Wiskunde* (2017, p. 247)

of Euler's formula $F + V - E = 2$ that states the relation between the number of faces, vertices, and edges of a (convex) solid. The story ends with an actual stamp from East-Germany with the face of Euler on it, some information on his life and work and the explanation of the letters used in the original formula, and finally two exercises to practice with the formula. In terms of the formats we defined earlier, this story contains two snippets (on Plato and on Euler's formula) and most of the other exercises are designed according to the psychological genetic principle.

The cognitive level may vary throughout the story. When we focus on the page shown above, exercise 10a can be labeled as cognitive demand level 2. Students have to look at the different platonic solids and simply count the number of vertices, faces, and edges. If they don't want to count them all, they might come up with a shortcut, and then voluntarily raise their own level of mathematical thinking. This is actually how a lot of mathematics is developed historically: the search for shortcuts or practical solutions to extensive problems. In exercises 10b and c, the students have to check if a certain rule, which is formulated in words, applies to all of the solids in the table. In exercise 11, this same rule is formulated with variables. The letters used in the formula are explained in text next to the picture of the German stamp. Because of the particular position of this picture and the accompanying text, this is in fact not a stamp according to our definition in Table 1 but a snippet. The level of cognitive demand in exercises 10bc and 11 is getting up to 3. Finally, the students have to think about terminology in exercise 12a and check the formula again in 12b. This will require a bit more mathematical thinking due to the semiregular shape of the solids, so we classify this as level 3 as well.

Although this story has a lot of potential for cognitive demand level 4, it is not accomplished through the exercises as formulated by the authors. For instance, the discovery of Euler's formula by students themselves is a missed opportunity. This could be asked as replacing exercises 10b and c, possibly with a question such as "can you spot patterns within this table." Another option for scaffolding is the standard pedagogical activity in mathematics that is called the ink-spot exercise, where part of a table, formula, or calculation is covered with a colored blurb and the students are asked to figure out what was written underneath. If the students were to discover, formulate, and check Euler's rule themselves, this would definitely account for cognitive demand level 4.

5 Discussion

In this article, we have discussed four formats to describe ways in which HoM can be integrated in curriculum materials, instructions, and other classroom activities. It was demonstrated that HoM can be applied as an effective tool to raise the level of mathematical thinking of students by combining the different formats to the four levels of cognitive demand by Stein and Lane (1996). Our so-called 4S-model not only offers the opportunity to categorize in a detailed manner different forms in which HoM can be integrated in mathematics education, but also to design them. For those purposes, the model offers criteria in terms of size, content, location, and function to make a clear distinction between the different formats in relation to the level of mathematical thinking that is pursued. As such, the 4S-model can be an interesting tool for teachers or in-service training programs to use HoM more often and more effectively in education. In making the distinction between the different formats of the 4S-model, we have emphasized that format as such is not a criterion for the cognitive demand level of tasks related to processing the offered historical

content. The different formats can contain historical content of varying difficulty. This also implies that tasks may be difficult in terms of mathematical processing but undemanding from an historical point of view or vice versa.

The model can function as a lens to identify curriculum materials with a certain level of cognitive demand, but teachers can also use the model to design their own instructions and activities incorporating HoM at a desired level. Incorporating HoM obviously requires a certain amount of basic knowledge on the subject matter, but this is not enough. Furinghetti (2007) concluded from an experiment with prospective teachers designing mathematics lessons that two different modes of integrating HoM in the teacher training program affected their teaching sequences. The first mode is the evolutionary approach, where the prospective teachers learn the development of certain mathematical concepts over time. The second mode is called the situated approach, which consists of the use of primary sources: “(the prospective teacher) recovers the cognitive roots through the historical roots and avoids sticking only to the polished theory that comes at the end of the evolutionary process” (Furinghetti, 2007, p. 137). When a prospective teacher has experienced the use of primary sources on several occasions herself in mathematics class during teacher training (from a student’s perspective), this can raise awareness on the possibilities to use original sources in her own teaching practice.

In addition to knowledge of HoM, teachers also require knowledge of how to use HoM. The situated approach shows one possibility, but the implementation of primary sources and secondary sources can be very challenging and may require different teaching skills. The pedagogy involved in teaching parts of the humanities (e.g., language, history) is usually not part of the mathematics teacher program. Teachers also need to be taught explicitly how to use HoM in classrooms in different ways. We suggest to explore the effects of a third approach of integrating HoM in teacher training programs. We call this the *design-oriented* approach, where prospect-teachers learn to apply HoM in their own lesson plans. The 4S-model, together with the use of the genetic principle and of primary sources, can serve as an instruction model for teacher training programs for this approach. Prospective teachers can learn how to identify the four formats and practice designing activities for students with a higher cognitive demand level. They can use either the psychological genetic principle or the historical genetic principle and/or use primary sources. To accommodate the prospective teachers in doing this, they not only need to be instructed in how to use HoM, but in how to find and select appropriate historical content as well (such as reference material on HoM or primary sources).

Haggarty and Pepin (2002) concluded, based on an empirical study, that approximately half of mathematics teachers rely entirely on the textbook and do not use any other sources when preparing classroom activities. When teachers are this bound to their textbooks, it requires curriculum materials that have HoM already integrated in them, in order to enable them to use it. The presence of HoM in Dutch textbooks is not superfluous. This tentative quantitative conclusion can be drawn as well from several international studies, listed by Schorcht (2018) in his article on HoM in German textbooks. The 4S-model can be used for a detailed analysis of textbooks and other curriculum materials, both quantitatively and qualitatively. Mathematics textbooks authors can even actively use the 4S-model themselves to write exercises and chapters including HoM and higher levels of cognitive demand. Attempts to make changes in mathematics teaching rely heavily on revision of methods/ learning materials, according to Remillard (2005).

However, Davis et al. (2016) state that mathematics teachers and science teachers tend to systematically reduce the cognitive level of high-level tasks in curriculum materials. The teacher can break up tasks into smaller subtasks that are simpler and require less

mathematical thinking and planning. The teacher can take over the thinking of students by telling them how to solve the problem, or by focusing on known routines and procedures (Stein et al., 1996). Reasons for doing so can differ from lack of (pedagogical) knowledge, to available time, or problems with classroom management. These reasons resemble some of the objections against the incorporation of HoM in mathematics classrooms, as listed by Clark et al. (2018, p. 7). We believe that the 4S-model, implemented in teacher training programs and supported as described earlier, can be considered as a concrete tool for (re)designing tasks. The use of this model will not enable teachers to overcome all of the difficulties in reaching a high level of cognitive demand, but it will help them in analyzing, assessing, and selecting or designing mathematical tasks for their classroom practices. Further research is needed to determine how a high cognitive level of tasks that incorporate HoM can be maintained throughout implementation.

The genetic principle, which we identified as a useful design principle for improving the level of cognitive demand, can bring along dilemmas for mathematics educators (Fried, 2014), (Furinghetti & Radford, 2008). Teachers are not historians and have other legitimate concerns that might even be conflicting. Freudenthal (1973) states that the use of HoM can make learning mathematics more difficult for some learners, since it can be more demanding not only mathematically but otherwise cognitively challenging as well. Nevertheless, he mostly sees opportunities for students to be challenged more. In addition, another important aspect of Freudenthal's didactical vision was to teach mathematics as a human activity. The use of HoM can play an important role in this as well. This is a possible aspect of the use of HoM that could serve as a welcome positive effect, in addition to raising the level of mathematical thinking. Students can obtain more motivation and develop a more positive attitude towards mathematics, for instance when they experience that the development of mathematics has been a process of trial and error, or if they become acquainted with more practical applications from the past of the seemingly abstract mathematics they learn in school today.

In order to raise the level of mathematical thinking, we are in pursuit of a more fundamental role for HoM in both teacher training and mathematics education. For this, we are inspired by the title of the ICME-13 monograph "Mathematics, Education and History: Towards a Harmonious Partnership" (Clark et al., 2018). With the 4S-model, we endeavor to contribute to the development of this partnership, step by step, from speck to story.

Author contribution Approved by all co-authors.

Data availability Not applicable.

Code availability Not applicable.

Declarations

Conflict of interest The authors declare no competing interests.

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